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Bianchi Identities, Electromagnetic Waves, and Charge Conservation in the $P(4)$ Theory of Gravitation and Electromagnetism

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The Bianchi identities for the $P(4) = O(1,3) \otimes \mathbb{R}^{4*}$ theory of gravitation and electromagnetism are decomposed into the standard $O(1,3)$ Riemannian Bianchi identity plus an additional $\mathbb{R}^{4*}$ component. When combined with the Einstein–Maxwell affine field equations the $\mathbb{R}^{4*}$ components of the $P(4)$ Bianchi identities imply conservation of magnetic charge and the wave equation for the Maxwell field strength tensor. These results are analyzed in light of the special geometrical postulates of the $P(4)$ theory. We show that our development is the analog of the manner in which the Riemannian Bianchi identities, when combined with Einstein's field equations, imply conservation of stress-energy-momentum and the wave equation for the Lanczos H-tensor.

1. INTRODUCTION

A fundamental feature of the general theory of relativity is that by combining Einstein's field equations with the doubly contracted Riemannian Bianchi identity one obtains an expression for conservation of the stress-energy-momentum tensor. Thus, as a consequence of the geometrical model of stress-energy-momentum, one need not inquire under what conditions the stress-energy-momentum tensor is conserved or what restrictions must be required to make it so. The stress-energy tensor is conserved automatically as a consequence of the *geometry of spacetime*. A second fundamental but perhaps less well-known feature of general relativity is that when the Einstein field equations are combined with the singly

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contracted Riemannian Bianchi identity one obtains a wave equation for the rank-three tensor $H^{\lambda}_{\mu\nu}$ introduced by Lanczos and which bears his name (Lanczos, 1962). Hence this wave equation may also be considered a consequence of the geometry of spacetime. The fact that these two fundamental features of general relativity follow from the Riemannian Bianchi identities is characteristic of the geometric spirit of Einstein's theory.

In this paper we shall consider the Bianchi identities for a geometry which is more general than the Riemannian geometry of Einstein's theory, namely the affine geometry of the $P(4) = O(1, 3) \otimes \mathbb{R}^{4*}$ theory of gravitation and electromagnetism (Norris, 1985, 1991; Kheyfets and Norris, 1988; Chilton and Norris, 1992). We will show that in the $P(4)$ theory one obtains a conservation law and a wave equation for the electromagnetic field that parallels those results mentioned above for the gravitational field, and that these new results also follow from the translational component of the $P(4)$ Bianchi identity.

The essential new idea in the $P(4)$ theory is to model the 4-momentum spaces of classical charged particles as affine spaces rather than linear vector spaces. In order to treat affine vectors on the same footing with vectors one must replace the $O(1, 3)$ Riemannian geometry of general relativity with $P(4) = O(1, 3) \otimes \mathbb{R}^{4*}$ affine geometry. In the resulting affine theory one obtains the Lorentz force law as the equation for an affine 4-momentum geodesic and, moreover, the Maxwell field equations are geometrized in terms of the $\mathbb{R}^{4*}$ component of the $P(4)$ curvature. The Bianchi identities of the $P(4)$ curvature contain in addition to the Riemannian Bianchi identities an additional component for the $\mathbb{R}^{4*}$ curvature. It is our purpose in this paper to decompose these additional identities and to analyze their physical content in light of the Einstein–Maxwell affine field equations which are presented in Section 2. The emphasis of the paper is not only on the relations obtained in this manner, namely an expression for conservation of magnetic charge and the standard wave equation for the Maxwell field strength tensor mentioned above, but also on the fact that these relations occur by virtue of identities in the extended geometry and thereby achieve a more elevated position in the $P(4)$ theory. Also, there are a number of structural parallels between the $P(4)$ theory of electromagnetism and general relativity which we shall point out in the course of the paper.

Section 2 begins with a brief review of the $P(4)$ theory. For a more complete description of the details the reader is referred to earlier works (Norris, 1985, 1991; Kheyfets and Norris, 1988; Chilton and Norris, 1992). The bundle structure is described with particular emphasis on how basic quantities transform under translational gauge changes. The translational degrees of freedom are used to model the 4-momentum spaces of classical charged particles as four-dimensional affine spaces. We are thereby enabled to model the Lorentz force law as the equation for an affine 4-momentum geodesic and in the process we identify the $\mathbb{R}^{4*}$ part of the $P(4)$ connection as the negative of the Maxwell field strength tensor. This identification allows the geometrization of the source-free Einstein–Maxwell equations which are then extended to include sources.

Section 3 is primarily concerned with the decomposition of the $\mathbb{R}^{4*}$ component of the $P(4)$ Bianchi identities. First the $P(4) = O(1,3) \otimes \mathbb{R}^{4*}$ Bianchi identity on a $P(4)$ principal fiber bundle is pulled back to the orthonormal frame bundle OM using the canonical embedding map of OM into AM. The pullbacks of the $P(4)$ curvature and connection forms are then decomposed into their $O(1,3)$ and $\mathbb{R}^{4*}$ components. The $\mathbb{R}^{4*}$ component of the pullback of the $P(4)$ connection is then further decomposed using the results of Norris *et al.* (1980). The resulting equation is then expressed on spacetime in component form relative to some general coordinated gauge. This equation is the $\mathbb{R}^{4*}$ component of the $P(4)$ Bianchi identity. We then form two contractions of this equation: one contraction of the equation in standard form and a second contraction of the equation in dual form. In Section 4, we consider the contraction of the $\mathbb{R}^{4*}$ component of the Bianchi identity in dual form in light of the Einstein–Maxwell affine field equations with sources and find that conservation of magnetic charge is implied as a consequence of the extended geometry. A parallel is then drawn between this derivation and the analogous derivation of conservation of stress-energy-momentum in general relativity. An alternate interpretation is then suggested which depends on a different identification of the $\mathbb{R}^{4*}$ component of the $P(4)$ connection. This identification leads to an expression for the conservation of electric (rather than of magnetic) charge as a consequence of the geometry of spacetime. The effect of this alternative, however, is that the Lorentz force law is generalized.

In Section 5 we show that when the $\mathbb{R}^{4*}$ component of the $P(4)$ Bianchi identity in standard form is combined with the Einstein–Maxwell affiine field equations one obtains the wave equation for the Maxwell field strength tensor in a curved spacetime as an identity of the $P(4)$ geometry. It is shown that 4-momentum is transferred from event to event in spacetime by the production of ripples in the affine vector field $\hat{0}$ which defines the local zeros of affine 4-momentum. Section 6 is devoted to a comparison between the electromagnetic wave equation as derived in the $P(4)$ theory and a similar derivation of a wave equation for the Lanczos tensor mentioned above. With this in mind a decomposition of the $\mathbb{R}^{4*}$ curvature is given which is similar in form to the decomposition of the Riemannian curvature tensor into the Weyl curvature tensor and terms involving only the Ricci tensor, the scalar curvature, and the metric.

A discussion of our results and conclusions are presented in Section 7. Also included in this section is a table which we use to make a structural comparison between general relativity, $P(4)$ electrodynamics, and standard $U(1)$ electrodynamics.

2. THE $P(4)$ THEORY OF ELECTROMAGNETISM AND GRAVITY

The geometrical arena of the $P(4)$ theory of gravitation and electromagnetism is the modified affine frame bundle, $\hat{A}M$, over a four-dimensional spacetime manifold M. Elements of $\hat{A}M$ are triples $(p, e_\mu, \hat{t})$, where $p \in M$, $(e_{\mu=1,2,3,4})$ is a linear frame at p, and $\hat{t}$ is an affine cotangent vector, the "origin" of the frame at p. This modification[3] is necessary because we wish to model the 4-momentum spaces of charged particles as affine spaces[4] and the 4-momentum is fundamentally a covector rather than a vector. The structure group of $\hat{A}M$ is the affine group $\hat{A}(4) = Gl(4) \otimes \mathbb{R}^{4*}$ with group multiplication

$$(A_1, \xi_1) \cdot (A_2, \xi_2) = (A_1 A_2, \xi_1 \cdot A_2 + \xi_2)$$

for all $(A_1, \xi_1), (A_2, \xi_2) \in \hat{A}(4)$ (Norris, 1991). Since $\hat{A}M$ is bundle isomorphic to AM, the standard affine frame bundle, we shall simplify the terminology and notation by referring to $\hat{A}M$ as the affine frame bundle of M and denote it by AM. Moreover, we will denote by $P(4)$ the Poincaré subgroup $O(1, 3) \otimes \mathbb{R}^{4*}$ of $\hat{A}(4)$.

AM is a principal fiber bundle over LM with standard fiber $\mathbb{R}^{4*}$. We shall refer to sections of AM over LM as translational gauges. A translational gauge can be thought of, therefore, as a choice of origin for local 4-momentum affine frames on M. It can be shown that translational gauges are in one-to-one correspondence with covector fields on M (Norris, 1991).

A generalized affine connection on AM can always be specified (Kobayashi and Nomizu, 1963; Norris *et al.*, 1980) by a pair $(\Gamma, {}^{\hat{t}}K)$ on spacetime, where Γ is a linear connection and ${}^{\hat{t}}K$ is a covariant vector valued one-form, where the left superscript indicates that K is represented in the $\hat{t}$ translational gauge. If the linear connection Γ is the Riemannian connection Γ_g of the metric tensor g, then the pair is said to represent a $P(4)$ connection. Furthermore, the pair $(\Gamma_g, {}^{\hat{t}}K)$ may be used to construct

[3] Ordinarily, the affine frame bundle AM is the set of triples $(p, e_i, \hat{t})$, where $\hat{t}$ is an affine tangent vector. The structure group of this bundle is $A(4) = Gl(4) \otimes \mathbb{R}^4$ (Kobayashi and Nomizu, 1963).

[4] An affine space (Dodson and Poston, 1977) is a triple (S, V, δ) where S is a set, V a vector space, and $\delta: S \times S \to V$ such that, for $\hat{x}, \hat{y}, \hat{z} \in S$, (1) $\delta(\hat{x}, \hat{y}) + \delta(\hat{y}, \hat{z}) = \delta(\hat{x}, \hat{z})$ and (2) for all $\hat{x} \in S$, the map $\delta_{\hat{x}}(\hat{y}) = \delta(\hat{y}, \hat{x})$ is a bijection.

the pair $(R, {}^{\hat{t}}\Phi)$, where R is the Riemannian curvature and ${}^{\hat{t}}\Phi$ is a covariant vector-valued 2-form on spacetime, the *affine* or $\mathbb{R}^{4*}$ component of the $P(4)$ *curvature*. Its components are defined by[5]

$$ {}^{\hat{t}}\Phi_{\mu\nu\lambda} = \nabla_\mu\, {}^{\hat{t}}K_{\lambda\nu} - \nabla_\nu\, {}^{\hat{t}}K_{\lambda\mu} \tag{2.1} $$

Under a translational gauge transformation, $\hat{t} \to \hat{x} = \hat{t} \oplus \vec{s}$, the $\mathbb{R}^{4*}$ component of the connection transforms as[6]

$$ {}^{\hat{t}\oplus\vec{s}}K_{\mu\nu} = {}^{\hat{t}}K_{\mu\nu} + \nabla_\mu s_\nu \tag{2.2} $$

and therefore, under the same transformation, we have

$$ {}^{\hat{t}\oplus\vec{s}}\Phi_{\mu\nu\lambda} = {}^{\hat{t}}\Phi_{\mu\nu\lambda} - R_{\mu\nu\lambda}{}^{\sigma}\, s_\sigma \tag{2.3} $$

If we define the contraction

$$ {}^{\hat{t}}\Phi_\mu \overset{\text{def}}{=} g^{\nu\lambda}({}^{\hat{t}}\Phi_{\mu\nu\lambda}) \tag{2.4} $$

then we obtain for it, from equation (2.3), the transformation law

$$ {}^{\hat{t}\oplus\vec{s}}\Phi_\mu = {}^{\hat{t}}\Phi_\mu - R_\mu{}^\nu\, s_\nu \tag{2.5} $$

Physically, we shall model the local 4-momentum spaces of classical charged particles as four-dimensional affine spaces (Norris, 1985). In such a space the observed 4-momentum must always be expressed relative to some local zero of affine 4-momentum. By this we mean that the observed 4-momentum is a vector ${}^{\hat{\sigma}}\vec{\pi}$ such that $\hat{\pi} = \hat{\sigma} \oplus {}^{\hat{\sigma}}\vec{\pi}$, where $\hat{\pi}$ is the affine 4-momentum and $\hat{\sigma}$ is the local zero (i.e., reference) of affine 4-momentum. We assume that there exists a translational gauge $\hat{0}$ such that, at a point along its trajectory in a nonzero electromagnetic field, the observed 4-momentum of the charged particle is the same as that of an instantaneously comoving and freely falling uncharged particle. In other words, $\hat{\pi} = \hat{0} \oplus \vec{u}$, where $\vec{u}$ is the 4-momentum per unit mass of the uncharged reference particle. We call $\hat{0}$ the *zero translation gauge* (Norris, 1985).

In order to transport the local zero of 4-momentum, defined at any single event in spacetime, to other events in spacetime, we must utilize an *affine transport law* based on the *affine covariant derivative* constructed from the pair $(\Gamma_g, {}^{\hat{t}}K)$. If $\hat{\pi}$ is the affine 4-momentum of the charged particle, then we say $\hat{\pi}$ is *affinely parallel* along the trajectory iff $\hat{D}\hat{\pi}/Ds = 0$, where $\hat{D}/Ds$ is the affine directional covariant derivative along the path. Written

[5]For a linear connection with torsion ${}^{\hat{t}}\Phi_{\mu\nu\lambda} \overset{\text{def}}{=} \nabla_\mu({}^{\hat{t}}K_{\lambda\nu}) - \nabla_\nu({}^{\hat{t}}K_{\lambda\mu}) + S_{\mu\nu}{}^{\sigma}({}^{\hat{t}}K_{\sigma\lambda})$, where $S_{\mu\nu}{}^{\sigma} = \Gamma_{[\mu\nu]}{}^{\sigma}$.

[6]The notation $\hat{x} = \hat{y} \oplus \vec{s}$ means that $\vec{s} = \delta(\hat{y}, \hat{x}) = \delta_{\hat{x}}(\hat{y})$.

in the zero translational gauge, this definition becomes

$$\left(\frac{\hat{D}\hat{\pi}}{Ds}\right)^{\mu} = \frac{Du^{\mu}}{Ds} + \epsilon({}^{\hat{0}}K^{\mu}{}_{\lambda})u^{\lambda} = 0 \tag{2.6}$$

where D/Ds is the linear directional covariant derivative and ϵ is the electric charge-to-mass ratio of the particle. In order to be compatible with Riemannian geometry it can be shown (Norris, 1985), using the fact that $(d/ds)[\vec{u}\cdot\vec{u}]=0$, that ${}^{\hat{0}}K_{(\mu\nu)}=0$. Consequently, if we identify ${}^{\hat{0}}K$ with the negative of the electromagnetic field strength tensor we obtain the equation

$$\frac{Du^{\mu}}{Ds} - \epsilon F^{\mu}{}_{\nu}u^{\nu} = 0 \tag{2.7}$$

Thus in the $P(4)$ theory the Lorentz force law arises as the equation describing an *affine 4-momentum geodesic*.

In making the identification ${}^{\hat{0}}K = -F$, we have implicitly assumed that F is the field strength tensor of standard $U(1)$ electromagnetism; that F is the curl of a vector field A and therefore that magnetic charges are not allowed. On the other hand, the only requirement that we have for ${}^{\hat{0}}K$ is that ${}^{\hat{0}}K_{(\mu\nu)}=0$. Consequently, there is a more general form available for ${}^{\hat{0}}K$ which is based on two potentials (Cabibbo and Ferrari, 1962), namely

$${}^{\hat{0}}K_{\mu\nu} = -(F_{\mu\nu} + M^{*}_{\mu\nu}) \equiv -\hat{F}_{\mu\nu} \tag{2.8a}$$

where

$$F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu} \tag{2.8b}$$

$$M_{\mu\nu} = \nabla_{\mu}B_{\nu} - \nabla_{\nu}B_{\mu} \tag{2.8c}$$

M^{*} in (2.8a) denotes the Hodge dual of M.

The standard Lorentz force law is obtained from (2.6) as a special case whenever $M^{*}_{\mu\nu}=0$. On the other hand, if $M^{*}_{\mu\nu}\neq 0$, we may make the identification ${}^{\hat{0}}K = -\hat{F} = -(F+M^{*})$, but then $\hat{F}$ is no longer the field strength tensor of standard $U(1)$ electromagnetism. Instead $\hat{F}$ in general represents the electromagnetic fields produced by both electric and magnetic currents. If this is the case, one may regard equation (2.7) as a *generalized Lorentz force law* that governs the motion of an electrically charged test particle in a field produced by both electric and magnetic currents.

Based on either of these identifications we may write the source-free Einstein–Maxwell equations in terms of $P(4)$ quantities as (Norris, 1985)

$${}^{\hat{0}}\Phi_{\mu} = 0 \tag{2.9}$$

$${}^{\hat{0}}\Phi_{[\mu\nu\lambda]} = 0 \tag{2.10}$$

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = {}^{\hat{0}}K_{\mu\lambda}\,{}^{\hat{0}}K_{\nu}{}^{\lambda} - \tfrac{1}{4}g_{\mu\nu}\,{}^{\hat{0}}K_{\lambda\sigma}\,{}^{\hat{0}}K^{\lambda\sigma} \tag{2.11}$$

We note that equation (2.11) is translationally invariant even though it may appear that the right-hand side is not. We have previously defined (Chilton and Norris, 1992) a difference tensor P_{ij} which is translationally invariant. This tensor is defined by

$$P_{ij} \stackrel{\text{def}}{=} {}^{\hat{t}}K_{ij} - {}^{\hat{t}}\bar{K}_{ij} \equiv {}^{\hat{0}}K_{ij}$$

where ${}^{\hat{t}}\bar{K}_{ij}$ is a flat $\mathbb{R}^{4*}$ connection with the property that ${}^{\hat{0}}\bar{K}_{ij} \equiv 0$. Thus (2.11) may also be written as

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = P_{\mu\lambda}P_{\nu}{}^{\lambda} - \tfrac{1}{4}g_{\mu\nu}P_{\lambda\sigma}P^{\lambda\sigma}$$

which is clearly translationally invariant.

These equations may be extended to include matter sources and both electric and magnetic currents as follows:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = {}^{\hat{0}}K_{\mu\lambda}\,{}^{\hat{0}}K_{\nu}{}^{\lambda} - \tfrac{1}{4}g_{\mu\nu}\,{}^{\hat{0}}K_{\lambda\sigma}\,{}^{\hat{0}}K^{\lambda\sigma} + T_{\mu\nu} \tag{2.12}$$

$${}^{\hat{0}}\Phi_{\mu} = -J_{\mu} \tag{2.13}$$

$${}^{\hat{0}}\Phi^{*}_{\mu} = -J^{*}_{\mu} \tag{2.14}$$

where $T_{\mu\nu}$, J_{μ}, and J^{*}_{μ} represent the nonelectromagnetic parts of the stress-energy-momentum tensor and the electric and magnetic current densities, respectively, and where we have defined

$${}^{\hat{0}}\Phi^{*}_{\mu} = \tfrac{1}{2}\epsilon_{\mu}{}^{\alpha\beta\gamma}\,{}^{\hat{0}}\Phi_{\alpha\beta\gamma} \tag{2.15}$$

Note that if we make the identification (2.8) where $M^{*}_{\mu\nu} \neq 0$, then J^{*}_{μ} need not be identically zero as required by the Bianchi identity of standard $U(1)$ electrodynamics. We refer to equations (2.12)–(2.14) as the Einstein–Maxwell affine field equations with sources. Recently it has been shown (Chilton and Norris, 1992) that these equations are derivable from a $P(4)$ variational principle.

3. THE $\mathbb{R}^{4*}$ COMPONENT OF THE *P*(4) BIANCHI IDENTITY AND ITS CONTRACTIONS

The $P(4)$ theory may be formulated directly on a subbundle of AM, the orthonormal affine frame bundle AOM, a principal bundle with standard fiber $P(4) = O(1, 3) \otimes \mathbb{R}^{4*}$. Given a connection $\hat{\omega}$ on this bundle, we may write the Bianchi identity as

$$\hat{D}\hat{\Omega} = d\hat{\Omega} + \hat{\omega} \wedge \hat{\Omega} \equiv 0 \tag{3.1}$$

where $\hat{\Omega}$ is the curvature with respect to the $P(4)$ connection. In order to identify the linear and translational components of objects, we pull them down to the orthonormal frame bundle over spacetime using the canonical embedding map $\gamma: LM \to AM$ defined by $\gamma(p, e_\mu) = (p, e_\mu, \hat{0})$, and split $\gamma^*(\hat{\omega})$ and $\gamma^*(\hat{\Omega})$ into $O(1,3)$ and $\mathbb{R}^{4*}$ components (Kobayashi and Nomizu, 1963). We find

$$\gamma^*(\hat{\omega}) = \omega_L + \omega_T \tag{3.2}$$

$$\gamma^*(\hat{\Omega}) = \Omega_L + \Omega_T \tag{3.3}$$

where the subscripts L and T refer to the linear $[O(1,3)]$ and translational $(\mathbb{R}^{4*})$ parts, respectively. If these are substituted into the pullback under γ of equation (3.1), we obtain

$$\gamma^*(\hat{D}\hat{\Omega}) = D^L\Omega_L + D^L\Omega_T + \omega_T \wedge \Omega_L \equiv 0 \tag{3.4}$$

where D^L means covariant differentiation with respect to the linear component of the connection, ω_L. Note that of the three terms on the right-hand side of equation (3.4), the only term which is an $o(1,3)$-valued 3-form is the term $D^L\Omega_L$, while the other two terms are $\mathbb{R}^{4*}$-valued 3-forms. Consequently, (3.4) implies not only the Riemannian Bianchi identity $D^L\Omega_L \equiv 0$, but also

$$D^L\Omega_T + \omega_T \wedge \Omega_L \equiv 0 \tag{3.5}$$

Equation (3.5) is the $\mathbb{R}^{4*}$ component of the $P(4)$ Bianchi identity, which we will refer to as the $\mathbb{R}^{4*}$ Bianchi identity.

It has been shown (Norris *et al.*, 1980) that on the orthonormal frame bundle, the $\mathbb{R}^{4*}$ component of the $P(4)$ connection may be further decomposed as

$$\omega_T = \rho\theta + \tau \tag{3.6}$$

where ρ is a scalar field, θ is the soldering 1-form on the frame bundle, and τ is an $\mathbb{R}^{4*}$-valued tensorial 1-form, which is uniquely related to a trace-free type $(0, 2)$ tensor field on spacetime. This decomposition is obtained by noting that ω_T corresponds to a type $(0, 2)$ tensor field on spacetime. The $\rho\theta$ and τ terms correspond to the trace and trace-free parts of this tensor field, respectively, with respect to the spacetime metric tensor. When (3.6) is substituted into the structure equation for the $\mathbb{R}^{4*}$ curvature, we find that

$$\Omega_T = \rho\Theta + d\rho \wedge \theta + D^L\tau \tag{3.7}$$

where Θ is the torsion 2-form on the frame bundle. Finally, when (3.6) and (3.7) are inserted into (3.5) we obtain the identity

$$D^L\Omega_T + \tau \wedge \Omega_L + \rho D^L\Theta + \rho\Theta \wedge \Omega_L \equiv 0 \tag{3.8}$$

In this paper we shall consider only the case when both ρ and Θ are zero and therefore (3.8) reduces to

$$D^L\Omega_T + \tau \wedge \Omega_L \equiv 0 \tag{3.9}$$

On spacetime, in a general coordinate gauge, this equation can be written in component form as

$$\nabla_{[\mu}(^{\hat{\imath}}\Phi)_{\alpha\beta]}{}^{\gamma} - R_{[\mu\alpha|}{}^{\sigma\gamma}(^{\hat{\imath}}K)_{\sigma|\beta]} = 0 \tag{3.10}$$

Alternatively, equation (3.10) may be written in dual form as

$$\nabla_{\mu}(^{\hat{\imath}}\Phi)^{\overset{*}{\mu\nu}\lambda} + R^{\overset{*}{\mu\nu}\lambda\sigma}(^{\hat{\imath}}K)_{\sigma\mu} = 0 \tag{3.11}$$

where we have defined

$$^{\hat{\imath}}\Phi^{\overset{*}{\mu\nu}\lambda} \overset{\text{def}}{=} \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}(^{\hat{\imath}}\Phi)_{\alpha\beta}{}^{\lambda} \tag{3.12}$$

and

$$R^{\overset{*}{\mu\nu}\lambda\sigma} \overset{\text{def}}{=} \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}R_{\alpha\beta}{}^{\lambda\sigma} \tag{3.13}$$

Since the $P(4)$ Bianchi identity involves the affine covariant derivative of affine tensorial fields, the resulting object is tensorial and therefore translationally invariant. Hence the Bianchi identity and the equations that result from it must be translationally invariant, as can be readily verified by the substitution of the transformation equations (2.2) and (2.3) into equations (3.10) and (3.11). Consequently we may express equations (3.10) and (3.11) in the $\hat{0}$ translational gauge with complete generality as follows:

$$\nabla_{[\mu}(^{\hat{0}}\Phi)_{\alpha\beta]}{}^{\gamma} + R_{[\mu\alpha}{}^{\sigma\gamma}\hat{F}_{|\sigma|\beta]} = 0 \tag{3.14}$$

and

$$\nabla_{\mu}(^{\hat{0}}\Phi)^{\overset{*}{\mu\nu}\lambda} + R^{\overset{*}{\mu\nu}\lambda\sigma}\hat{F}_{\mu\sigma} = 0 \tag{3.15}$$

We shall now consider contractions of these two equations.

If we contract equation (3.14) on the indices μ and γ, we obtain

$$\nabla^{\mu}\nabla_{\mu}\hat{F}_{\alpha\beta} - \tfrac{3}{2}\nabla^{\mu}(^{\hat{0}}\Phi_{[\alpha\beta\mu]}) - 2\nabla_{[\alpha}{}^{\hat{0}}\Phi_{\beta]} = R_{\alpha\sigma}\hat{F}^{\sigma}{}_{\beta} - R_{\beta\sigma}\hat{F}^{\sigma}{}_{\alpha} + R_{\alpha\beta\sigma}{}^{\mu}\hat{F}^{\sigma}{}_{\mu} \tag{3.16}$$

where we have used the identity

$$\nabla_{\mu}{}^{\hat{0}}\Phi_{\alpha\beta}{}^{\mu} = -\nabla^{\mu}\nabla_{\mu}\hat{F}_{\alpha\beta} - \tfrac{3}{2}\nabla^{\mu}(^{\hat{0}}\Phi_{[\alpha\mu\beta]})$$

On the other hand, contraction of equation (3.15) on the indices ν and λ results in the equation

$$\nabla^{\mu}(^{\hat{0}}\Phi^{*}_{\mu}) = 0 \tag{3.17}$$

The second term in equation (3.15) vanishes upon contraction on ν and λ due to the symmetries of the Riemannian curvature tensor.

4. CONSERVATION OF CHARGE IN THE $P(4)$ THEORY

When equation (2.14) is substituted into equation (3.17) we obtain

$$\nabla_{\mu} \overset{*}{J}{}^{\mu} \equiv 0 \tag{4.1}$$

Thus, in the $P(4)$ theory, magnetic charge is conserved as an identity of the geometry. On the other hand, conservation of electric charge has been shown (Chilton and Norris, 1992) to be a consequence of the symmetries of the $P(4)$ Lagrangian.

Conservation of magnetic charge in $P(4)$ electromagnetism therefore appears to be the analog of conservation of stress-energy-momentum in general relativity. While it is not possible to contract the $\mathbb{R}^{4*}$ component of the Bianchi identity [equation (3.10)] twice due to the fact that a second contraction must necessarily be on an antisymmetric pair of indices, one may still draw a parallel between the derivation of equation (4.1) and an alternative derivation (Synge, 1960) of the conservation of the stress-energy-momentum tensor in general relativity. Recall that if one forms the double dual curvature tensor,

$$*R^{\alpha\beta\gamma\delta} \overset{\text{def}}{=} \tfrac{1}{4}\epsilon^{\alpha\beta\mu\nu}\epsilon^{\gamma\delta\lambda k}R_{\mu\nu\lambda k} \tag{4.2}$$

then the Riemannian Bianchi identity may be written as

$$\nabla_{\alpha} *R^{\alpha\beta\gamma\delta} \equiv 0 \tag{4.3}$$

Contracting the indices β and γ gives

$$\nabla_{\alpha} *R^{\alpha\delta} = 0 \tag{4.4}$$

where

$$*R^{\alpha\delta} \overset{\text{def}}{=} g_{\beta\gamma} *R^{\alpha\beta\gamma\delta} = R^{\alpha\delta} - \tfrac{1}{2}g^{\alpha\delta}R \tag{4.5}$$

Using the Einstein field equations, one obtains

$$\nabla_{\alpha} T^{\alpha\delta} \equiv 0 \tag{4.6}$$

Thus the conservation of stress-energy-momentum in general relativity and the conservation of magnetic charge in the $P(4)$ theory both are obtained by a single contraction of the respective Bianchi identities in dual form.

It is important to note that the fact that we have obtained a conservation law for magnetic charge is ultimately due to the fact that we originally identified the $\mathbb{R}^{4*}$ connection component in the $\hat{0}$ gauge with the negative of the generalized Maxwell field strength tensor $\hat{F} = F + M^*$ [see (2.8)]. This was done in order to model the Lorentz force law of electrically charged particles as an affine 4-momentum geodesic. However, an alternate approach is available.

Let us define

$$^{\hat{0}}K_{\mu\nu} \overset{\text{def}}{=} -(F_{\mu\nu} + \alpha F^{*}_{\mu\nu}) \tag{4.7}$$

where $F_{\mu\nu}$ is the standard $U(1)$ field strength tensor and α is assumed to be a small constant. Note that this is equivalent to setting $M_{\mu\nu} = \alpha F_{\mu\nu}$ in equation (2.8a). This is a very special case of (2.8), since generally $F_{\mu\nu}$ and $M^{*}_{\mu\nu}$ are independent. In this special case we have

$$\begin{aligned} ^{\hat{0}}\Phi^{*\mu} &= \tfrac{1}{2}\epsilon^{\mu\alpha\beta\gamma}(^{\hat{0}}\Phi)_{\alpha\beta\gamma} \\ &= 2(\nabla_\beta \overset{*}{F}{}^{\mu\beta} - \alpha\nabla_\beta F^{\mu\beta}) \end{aligned}$$

However, $\nabla_\beta \overset{*}{F}{}^{\mu\beta} = 0$ and therefore

$$\nabla_\mu \, ^{\hat{0}}\Phi^{*\mu} = -\alpha\nabla_\mu\nabla_\beta F^{\mu\beta} = -\alpha\nabla_\mu J^\mu = 0$$

We see therefore that if we make the identification (4.7) we obtain conservation of electric charge, rather than magnetic charge, as a consequence of the geometry. This is not completely satisfactory, however, since when one inserts (4.7) into equation (2.6), one obtains

$$\frac{Du^\mu}{Ds} = \epsilon(F^\mu{}_\nu + \alpha F^{\mu *}{}_\nu)u^\nu \tag{4.8}$$

The effect of this alternative approach therefore is that the Lorentz force law is generalized. Equation (4.8) describes the motion of a particle with an electric charge-to-mass ratio ϵ and a magnetic charge-to-mass ratio $\alpha\epsilon$ (Schwinger, 1969).

5. ELECTROMAGNETIC WAVES IN THE P(4) THEORY

In order to deduce the physical significance of equation (3.16), we assume the field equations (2.13) and (2.14) and we also assume that $\overset{*}{J}{}^\mu = 0$, which implies that $^{\hat{0}}\Phi_{[\alpha\beta\mu]} = 0$. Equation (3.16) therefore reduces to

$$\Box\hat{F}_{\alpha\beta} - R_{\alpha\sigma}\hat{F}^\sigma{}_\beta + R_{\beta\sigma}\hat{F}^\sigma{}_\alpha - R_{\alpha\beta\sigma}{}^\mu\hat{F}^\sigma{}_\mu = 2\nabla_{[\alpha}J_{\beta]} \tag{5.1}$$

This is the standard wave equation for the Maxwell field strength tensor in a curved spacetime, and thus we see that electromagnetic waves occur as an identity of $P(4)$ geometry. Note that in contrast to the usual version of the electromagnetic wave equation, that is, in terms of the vector potential, this wave equation is expressed in terms of the manifestly physical **E** and **B** fields.

Hence the possibility of transferring 4-momentum from one point to another point via electromagnetic waves occurs as an intrinsic property of the geometry of spacetime in the $P(4)$ theory. In the $P(4)$ theory an electromagnetic wave appears as a "rippling" of the affine vector field $\hat{0}$ which represents the zero or reference of affine 4-momentum. In order to see that this rippling does in fact occur due to the passage of an electromagnetic wave, one must be able to compare the field at neighboring points in spacetime. This may be accomplished by utilizing the electromagnetic affine differential transport law first introduced by Norris (1985).

If the $\hat{0}$ affine vector field is defined at a point p_1 in Minkowski spacetime, then one may transport it to a neighboring point p_2 along any curve γ which joins p_1 and p_2. The transported zero of 4-momentum will be an affine vector $\hat{z}$ defined at p_2 given by

$$\hat{z}(p_2) = \hat{0}(p_2) \oplus \left[-q \int_\gamma F^\lambda{}_\kappa \, dx^\kappa \right] \frac{\partial}{\partial x^\lambda}\bigg|_{p_2} \tag{5.2}$$

Alternatively one may write

$$0^\lambda(p_2) = -q \int_\gamma F^\lambda{}_\kappa \, dx^\kappa \tag{5.3}$$

where

$$0^\lambda(p_2) \stackrel{\text{def}}{=} [\delta(\hat{0}(p_2), \hat{z}(p_2))]^\lambda \tag{5.4}$$

is the total 4-impulse imparted to a charge q which is constrained to move along the path γ. The transported zero of affine 4-momentum is in general a path-dependent quantity that is path independent if and only if ${}^0\Phi_{\mu\nu\lambda} \equiv 0$.

To illustrate the rippling of the $\hat{0}$ field, we consider a simple example. Suppose an electromagnetic wave of the form

$$F^\lambda{}_\kappa = \bar{F}^\lambda{}_\kappa \sin[k_\mu x^\mu]$$

where $\bar{F}^\lambda{}_\kappa$ is a constant skew-symmetric tensor and k_μ is a null vector, passes the origin of the Minkowskian coordinate system, which we take to be the point $p_1 = (0, 0, 0, 0)$ where the $\hat{0}$ field is initially defined. Then we may integrate along the time line to any point p_2 of the form $p_2 = (t, 0, 0, 0)$ as follows:

$$\begin{aligned} \hat{z}(p_2) &= \hat{0}(p_2) \oplus \left[-q\bar{F}^\lambda{}_0 \int_0^t \sin(k_0 x^0) \, dx^0 \right] \frac{\partial}{\partial x^\lambda}\bigg|_{p_2} \\ &= \hat{0}(p_2) \oplus \left[q \frac{\bar{F}^\lambda{}_0}{k_0} \cos(k_0 t) \right] \frac{\partial}{\partial x^\lambda}\bigg|_{p_2} \end{aligned} \tag{5.5}$$

This result is clearly periodic. A free charged particle initially located at the origin will begin to oscillate due to the changing of the $\hat{0}$ affine vector field. In terms of the $P(4)$ theory, this is because a free charged particle must follow an affine 4-momentum geodesic ($\hat{D}\hat{\pi}/Ds = 0$) and since $\hat{\pi} = \hat{0} \oplus \vec{\pi}$, if

$\hat{0}$ changes, then $\vec{\pi}$, the observed 4-momentum, must change in just the proper amount to counterbalance the change in the $\hat{0}$ field. This may be compared with the Riemannian geodesic equation for free uncharged particles. In that case the components of the 4-velocity, defined relative to any linear frame, must change in just the proper amount to counterbalance the linear curvature-induced change in the reference frame.

The fact that the standard electromagnetic wave equation for the Maxwell field strength tensor occurs as a geometrical identity in the $P(4)$ theory gives the $P(4)$ theory one other feature not shared by general relativity, and that feature is the existence of a practical mechanism within the geometry of the theory which allows the determination of the null subspaces of local tangent spaces. Although general relativity predicts the existence of gravitational waves which propagate at the velocity of light, to date no such waves have been observed, and so, as a matter of practicality, one must appeal to a nongeometrical source of information, e.g., electromagnetic waves, to determine the null subspaces. In the $P(4)$ theory one still must observe electromagnetic waves in order to determine the null subspaces, but in the $P(4)$ theory the information is built into the geometry. Furthermore, once the null subspaces have been determined, one may determine the metric tensor of spacetime up to a constant conformal factor (Hawking and Ellis, 1973).

6. A COMPARISON WITH WAVES IN GENERAL RELATIVITY

In Section 5 we have shown that electromagnetic waves occur as an identity in the $P(4)$ geometry. As we have mentioned above, there is an analogous result in general relativity, namely the wave equation for the rank-3 tensor field $H_{\alpha\beta\gamma} = -H_{\beta\alpha\gamma}$ known as the Lanczos spin tensor. This tensor was shown by Lanczos (1962) to exist for all Riemannian spacetimes. Furthermore, it can be said to serve as a potential for the Weyl conformal curvature tensor $C_{\mu\nu\alpha\beta}$, where

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + g_{\mu[\alpha}R_{\beta]\nu} - g_{\nu[\alpha}R_{\beta]\mu} - \tfrac{1}{3}g_{\mu[\alpha}g_{\beta]\nu}R \tag{6.1}$$

Specifically, Lanczos showed that the Weyl conformal curvature tensor can be rewritten as

$$\begin{aligned} C_{\mu\nu\alpha\beta} = {} & \nabla_\beta H_{\mu\nu\alpha} - \nabla_\alpha H_{\mu\nu\beta} + \nabla_\nu H_{\alpha\beta\mu} - \nabla_\mu H_{\alpha\beta\nu} \\ & + \tfrac{1}{2}[g_{\mu\beta}(H_{\nu\alpha} + H_{\alpha\nu}) - g_{\nu\beta}(H_{\mu\alpha} + H_{\alpha\mu}) \\ & + g_{\alpha\nu}(H_{\beta\mu} + H_{\mu\beta}) - g_{\mu\alpha}(H_{\nu\beta} + H_{\beta\nu})] \end{aligned} \tag{6.2}$$

where we have introduced the notation $H_{\mu\lambda} = \nabla_\nu H_\mu{}^\nu{}_\lambda$. In the above relation we have also imposed the standard Lanczos "algebraic conditions," namely

$H_{[\alpha\beta\gamma]}=0$ and $H_{\alpha\beta}{}^{\alpha}=0$, as well as his "differential condition" $\nabla_\nu H_{\beta\gamma}{}^{\nu}=0$. The first contracted Riemannian Bianchi identities in conjunction with relation (6.2) leads to the following wave equation for $H_{\alpha\beta\gamma}$:

$$\begin{aligned}\Box H_{\alpha\beta\gamma} &+ R_{\alpha\mu\gamma\nu}H_{\beta}{}^{\nu\mu} - R_{\beta\mu\gamma\nu}H_{\alpha}{}^{\nu\mu} - R_{\alpha\beta\nu\mu}H_{\gamma}{}^{\nu\mu}\\ &+ (R_{\alpha\nu\mu\lambda}g_{\beta\gamma} - R_{\beta\nu\mu\lambda}g_{\alpha\gamma})H^{\nu\mu\lambda} - R_{\alpha}{}^{\nu}H_{\gamma\nu\beta} + R_{\beta}{}^{\nu}H_{\gamma\nu\alpha} - R_{\gamma}{}^{\nu}H_{\alpha\beta\nu}\\ &= \tfrac{1}{2}\nabla_\alpha(R_{\beta\gamma} - \tfrac{1}{6}g_{\beta\gamma}R) - \tfrac{1}{2}\nabla_\beta(R_{\alpha\gamma} - \tfrac{1}{6}g_{\alpha\gamma}R)\end{aligned} \tag{6.3}$$

The above wave equation for the Lanczos tensor is an identity of the Riemannian geometry. This geometrical identity has physical meaning only when field equations are assumed. Indeed, if the Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \tag{6.4}$$

is used in (6.3), the right-hand side can be rewritten as $\nabla_{[\alpha}J_{\beta]\gamma}$, where $J_{\beta\gamma} = T_{\beta\gamma} - 1/3g_{\beta\gamma}T$ represents the gravitational sources for the Lanczos spin tensor. Clearly these results are analogous to the contraction of the $\mathbb{R}^{4*}$ component of the $P(4)$ Bianchi identity as given in (3.16) and its corresponding reduction to a physically measurable quantity, namely (5.1), which arises when the Maxwell field equations are imposed. These wave equations arise from Bianchi identities and suggest a strong parallel between the Weyl and $\mathbb{R}^{4*}$ curvature tensors.

This analogy can be made more concrete. Recall that all contractions of the conformal curvature tensor vanish, which follows from the decomposition as given in (6.1). We can make a similar decomposition of the $\mathbb{R}^{4*}$ curvature. In a general translational gauge $\hat{t}$ we define

$$^{\hat{t}}C_{\alpha\beta}{}^{\gamma} \stackrel{\text{def}}{=} {}^{\hat{t}}\Phi_{\alpha\beta}{}^{\gamma} + \tfrac{2}{3}\delta^{\gamma}_{[\alpha}({}^{\hat{t}}\Phi)_{\beta]} - g^{\gamma\sigma}({}^{\hat{t}}\Phi)_{[\alpha\beta\sigma]} \tag{6.5}$$

As with the Weyl conformal curvature tensor, all traces of $^{\hat{t}}C_{\alpha\beta}{}^{\gamma}$ are zero. Furthermore, it satisfies $^{\hat{t}}C_{[\alpha\beta\gamma]}=0$. In the zero translational gauge we may write $^{\hat{0}}C_{\alpha\beta\gamma}$ as

$$\begin{aligned}^{\hat{0}}C_{\alpha\beta\gamma} = -\tfrac{1}{3}[&\nabla_\alpha\hat{F}_{\beta\gamma} - \nabla_\beta\hat{F}_{\alpha\gamma} + \nabla_\gamma\hat{F}_{\beta\alpha} - \nabla_\gamma\hat{F}_{\alpha\beta}\\ &+ g_{\gamma\alpha}\nabla_\sigma\hat{F}_{\beta}{}^{\sigma} - g_{\gamma\beta}\nabla_\sigma\hat{F}_{\alpha}{}^{\sigma}]\end{aligned} \tag{6.6}$$

which is similar in form to relation (6.2).

Thus the Bianchi identities lead to wave identities for the "potentials" $H_{\alpha\beta\gamma}$ and $^{\hat{0}}K_{\alpha\beta} = -\hat{F}_{\alpha\beta}$ of the Weyl and $\mathbb{R}^{4*}$ curvature tensors, respectively. These identities become physically significant once the appropriate field equations are imposed, namely, the Einstein and Maxwell equations, respectively. The above analysis implies that the $\mathbb{R}^{4*}$ component of the $P(4)$ connection $^{\hat{t}}K_{\alpha\beta}$ may play a role in $P(4)$ electrodynamics which is analogous to the role of the Lanczos tensor $H_{\alpha\beta}{}^{\gamma}$ in standard general relativity.

7. CONCLUSIONS AND DISCUSSION

The fundamental principle of the $P(4)$ theory of gravitation and electromagnetism is to model the energy-momentum spaces of classical charged particles as affine spaces. The immediate consequence of this generalization is the geometrization of the Lorentz force law as the geodesic equation of a generalized geometry for spacetime: $P(4)$ geometry. As a result of this generalization, more physical phenomena than before are recognized as fundamentally geometrical in character.

In this paper we have been concerned with those phenomena occurring by virtue of the Bianchi identities for that extended geometry beyond those inherent in the Riemannian Bianchi identities alone. These are, namely, electromagnetic waves and conservation of magnetic charge. In both cases we have demonstrated close structural parallels between $P(4)$ electromagnetism and general relativity with regard to the manner in which their respective identities are obtained.

In Section 3 we formed the contraction of the $\mathbb{R}^{4*}$ component of the $P(4)$ Bianchi identities in dual form. When this was combined with the Einstein–Maxwell affine field equations in Section 4, the result was an expression for the conservation of magnetic charge. The technique used to achieve this result is closely parallel to an alternative derivation of the conservation law for the stress-energy-momentum tensor in general relativity. With a slightly different identification of the $\mathbb{R}^{4*}$ component of the connection [equation (4.7)], the same technique was used to derive an expression for the conservation of electric charge. The identification, however, leads to a generalized Lorentz force law [equation (4.8)] that includes a magnetic charge term. In Section 3 we also formed a contraction of the $\mathbb{R}^{4*}$ component of the $P(4)$ Bianchi identity in undualed form, and this yielded in Section 5 the wave equation for the Maxwell field strength tensor when Maxwell's equations were assumed. In the $P(4)$ theory electromagnetic waves are interpreted as ripples in the $\hat{0}$ affine gauge field, which defines the local zeroes of affine 4-momentum. A free charged particle experiencing a fluctuation in its zeros of 4-momentum must begin to oscillate relative to inertial frames in order to follow an affine 4-momentum geodesic. It is in this manner that 4-momentum is transferred to charged particles by an electromagnetic wave.

Furthermore, consideration of the transmission of 4-momentum via electromagnetic waves in the context of the $P(4)$ theory serves to complete our philosophical view of tee information contained in the Bianchi identities. The Riemannian Bianchi identities tell us, via Poynting's theorem, that when the 4-momentum contained in an electromagnetic field is allowed to move through space, the change in the amount of 4-momentum contained in a volume of space is equal to the flux of 4-momentum across the boundary. The Riemannian Bianchi identities, however, neglect to inform us as to just how the 4-momentum is to be transported, and this is one service which the $\mathbb{R}^{4*}$ component of the $P(4)$ Bianchi identities performs: 4-momentum may be transferred via *electromagnetic waves*.

As we have demonstrated in Section 6, the analog of the electromagnetic wave equation in general relativity is the wave equation for the Lanczos H-tensor. Furthermore, we have shown a decomposition of the $\mathbb{R}^{4*}$ curvature which is similar to the decomposition of the Riemannian curvature tensor into the Weyl tensor and terms depending on the Ricci tensor, the scalar curvature, and the metric tensor. These observations may be very significant for the following reason: throughout this paper and others in the literature the structural similarities between $P(4)$ electrodynamics and general relativity have been exploited in order to clarify the interpretation of various aspects of $P(4)$ electrodynamics. In this case, as we have mentioned, it is the object associated with general relativity, that is, the Lanczos tensor, which is in need of illumination. It is thought by some that the Lanczos tensor may be fundamentally associated with gravitational radiation, and the parallels noted in this paper seem to bear out that interpretation. But is it truly the $\mathbb{R}^{4*}$ component of the $P(4)$ connection ${}^{\hat{\imath}}K_{\mu\nu}$ which is the analog of the Lanczos tensor? In order to be certain, one should produce the analog of the Lanczos tensor by the same method used by Lanczos: a variational principle. It is our intention to address this issue in future publications.

Throughout this paper we have pointed out parallels between general relativity and $P(4)$ electrodynamics. In conclusion, we wish to summarize these similarities and contrast them with the standard interpretation of electrodynamics as a $U(1)$ gauge theory. These results are summarized in Table I. Note that in both general relativity and $P(4)$ electrodynamics there is a three-tiered hierarchy of geometrical objects: the curvature is constructed from the connection and its derivatives and in certain cases, when the linear geometry is Riemannian or when the generalized affine connection is expressed in the $\hat{0}$ gauge, the connection may be expressed in terms of a potential or potentials and their derivatives. In contrast, the $U(1)$ connection A_μ is not constructable from a potential. Furthermore, for both general relativity and $P(4)$ electrodynamics, sources appear in the field equations at the level of the respective contracted curvatures, whereas in $U(1)$ electrodynamics, sources appear at one higher differential order. Thus we see that when viewed from the perspective of the differential structure of the $P(4)$ theory, gravity and electromagnetism seem very much alike, whereas $U(1)$ electromagnetism seems very different. In addition, $P(4)$ electrodynamics differs fundamentally from the standard $U(1)$ interpreta-

Table I

	General relativity	$P(4)$ Electromagnetism	$U(1)$ Electromagnetism
Curvature	$R_{\mu\nu\nu}{}^{\kappa} = \partial_\mu \{^{\kappa}_{\nu\lambda}\} - \partial_\nu \{^{\kappa}_{\mu\lambda}\} + \{^{k}_{\mu\rho}\}\{^{\rho}_{\nu\lambda}\} - \{^{k}_{\nu\rho}\}\{^{\rho}_{\mu\lambda}\}$	${}^{0}\Phi_{\mu\nu\lambda} = \nabla_\mu({}^{0}K)_{\lambda\nu} - \nabla_\nu({}^{0}K)_{\lambda\mu}$	$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$
Connection	$\{^{\lambda}_{\mu\nu}\} = \frac{1}{2} g^{\lambda\sigma}[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}]$	${}^{0}K_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + \epsilon_{\mu\nu}{}^{\lambda\kappa}\nabla_\lambda B_\kappa$	A_μ
Potentials	$g_{\mu\nu}$	A_μ, B_μ	Not applicable
Bianchi identity	$\nabla_{[\mu} R_{\nu\lambda]\sigma}{}^{\kappa} = 0$	$\nabla_{[\mu}({}^{0}\Phi)_{\alpha\beta]}{}^{\gamma} - R_{[\mu\alpha}{}^{\sigma\gamma}({}^{0}K)_{\lvert\sigma\rvert\beta]} = 0$	$\nabla_{[\mu} F_{\nu\lambda]} = 0$
Field equations with sources	$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$	${}^{0}\Phi_\mu = -J_\mu$, ${}^{0}\Phi^*_\mu = -J^*_\mu$	$\nabla^\lambda F_{\lambda\mu} = J_\mu$
Lagrangian	Linear in curvature	Linear in curvature	Quadratic in curvature
Contraction of Bianchi identity in dual form implies conservation law	Yes, $\nabla_\mu T^{\mu\nu} = 0$	Yes, $\nabla_\mu \overset{*}{J}{}^{\mu} = 0$	No, cannot be contracted
Contraction of Bianchi identity in undulated form gives wave equation	Yes, for $H_{\mu\nu}{}^{\lambda}$	Yes, for ${}^{0}K_{\mu\nu} = -F_{\mu\nu}$	No

tion of the electromagnetic field in that the $P(4)$ theory is capable of fully accommodating the two-potential formalism of Cabibbo and Ferrari (1962). Recall that not only do the two potentials in the Cabibbo–Ferrari theory each possess the $U(1)$ gauge freedom, but also they collectively possess the freedom of the so-called mixing transformation.

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